The Dynamics of the Falling Slinky
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1 Introduction

The slinky, a heavy helical spring of steel or plastic with a low coefficient of elasticity, is a delightful toy. When tipped over at the top of a flight of stairs, it falls end over end from step to step. It can be used to perform a variety of other tricks and also to demonstrate a range of wave phenomena.

The slinky was invented in 1943 by a naval engineer, Richard James, who was inspired by the behaviour of springs that he had accidently knocked over. The toy was an instant hit and many millions of slinkys have been sold.

Recent interest has focussed on the behaviour of a slinky dropped from a hanging state. It is surprising that the bottom of the falling body remains stationary while the top collapses, coil upon coil, until it reaches the bottom. Several videos of this phenomenon are available on YouTube [3].

2 The Equation of the Slinky

The slinky is a pre-tensioned spring: in its resting state, the coils are pressed together and a force is required to separate them. We denote the length at full compression by \( \ell_1 \) and the length for which the tension vanishes by \( \ell_0 \). However, the length \( \ell_0 \) is physically unattainable, as it is less than the fully compressed length \( \ell_1 \).

To describe the dynamics, we introduce a coordinate \( \zeta \) varying from 0 to 1 along the length of the spring, with the mass distributed uniformly with \( \zeta \). Thus, \( dM = M d\zeta \) is the mass of an element \( d\zeta \) and the total mass is \( \int_0^1 M d\zeta \).

We assume that the spring is confined to a vertical line. The height at a given point \( \zeta \) and time \( t \) is \( z(\zeta, t) \). The stretching factor is \( \partial z / \partial \zeta \) so, if spring constant is \( K \), the tension is \( K (\partial z / \partial \zeta - \ell_0) \).

In the sequel, we assume \( \ell_0 = 0 \).

The Lagrangian of a slinky is the difference between kinetic and potential energy:

\[
L = \int_0^1 \left[ \frac{1}{2} M \left( \frac{\partial z}{\partial t} \right)^2 - \frac{1}{2} K \left( \frac{\partial z}{\partial \zeta} \right)^2 - Mgz \right] d\zeta . \tag{1}
\]

From this we can immediately write the equation of motion

\[
M \frac{\partial^2 z}{\partial t^2} = K \frac{\partial^2 z}{\partial \zeta^2} - Mg . \tag{2}
\]

Equilibrium solution

We first consider the case where the spring is hanging vertically, suspended at one end. Denoting this solution by \( z_0(\zeta) \), (2) reduces to

\[
K \frac{d^2 z}{d\zeta^2} = g ,
\]

Defining \( \omega^2 = K/M \) and \( \lambda = g/\omega^2 = Mg/K \), the solution is

\[
z_0 = \frac{1}{2} \lambda \zeta^2 .
\]

We note that the length of the hanging spring is \( z_0(1) = \frac{1}{2} \lambda \). This contrasts with a mass \( M \) suspended from a massless spring of stiffness \( K \) and natural length \( \ell_0 = 0 \), which has an equilibrium length \( \lambda \).
Solution after release

The stretching of the spring is given by $\frac{\partial z}{\partial \zeta}$ and, after release, this must vanish at both ends as there is nothing to balance the tension. Thus, the boundary conditions are

$$\frac{\partial z}{\partial \zeta} = 0 \text{ at } \zeta = 0 \quad \text{and} \quad \frac{\partial z}{\partial \zeta} = 0 \text{ at } \zeta = 1.$$  

When the slinky is considered as a whole, it is clear that the centre of mass must have a downward acceleration $g$, as for any freely falling body. We seek a solution comprising a term representing the fall of the centre of mass and a deviation depending on position within the slinky:

$$z(\zeta, t) = -\frac{1}{2}gt^2 + z'(\zeta, t).$$

The deviation $z'$ satisfies the homogeneous wave equation

$$\frac{\partial^2 z'}{\partial t^2} = \omega^2 \frac{\partial^2 z'}{\partial \zeta^2} \quad (3)$$

and the boundary conditions

$$\frac{\partial z'}{\partial \zeta} = 0 \text{ at } \zeta = 0 \quad \text{and} \quad \frac{\partial z'}{\partial \zeta} = -\lambda \text{ at } \zeta = 1. \quad (4)$$

The initial conditions are

$$z' = \frac{1}{2}\lambda \zeta^2 \quad \text{and} \quad \frac{\partial z'}{\partial t} = 0 \text{ at } t = 0. \quad (5)$$

The system $(3), (4)$ and $(5)$ completely defines the problem.

Character of the solution

It is possible to write the solution of the system $(3–5)$ explicitly using d’Alembert’s solution [2]. However, it is more illuminating to consider the solution in a qualitative way. Before release at $t = 0$, the solution is $z = z_0(\zeta)$. At time $t = 0$, the removal of the constraint at $\zeta = 1$ results in a discontinuity: the slinky is no longer in balance, and a wave is generated. This propagates at speed $c = L\omega$ down the spring, taking time $\tau = \sqrt{\frac{M}{K}}$ to travel from one end to the other. At any stage, there is a wave front, below which the slinky remains undisturbed until the front reaches it.

At the same time, the uppermost coils of the slinky accelerate downwards. Their rate of acceleration is greater than the free-fall rate $g$, as the downward tension augments the force of gravity. The slinky collapses downward from the top.

Numerical solution

The system $(3–5)$ is integrated numerically from $t = 0$ to the time of total collapse, $t = \tau$. We choose a spring mass $M = 0.24$ kg and length $L = 1.7$ m corresponding to a typical slinky. With $g = 9.81$ m s$^{-2}$, this implies $k = 0.69$ kg s$^{-2}$, $\omega = 1.70$ s$^{-1}$, $\lambda = 3.40$ m and $\tau = 0.59$ s.

In Fig. 1 we show the numerical solution $z(\zeta, t)$ for $t \in \{0.2\tau, 0.4\tau, 0.6\tau, 0.8\tau\}$. The solution is constrained so that $z(\zeta_1) \geq z(\zeta_2)$ when $\zeta_1 \geq \zeta_2$. This is in recognition of the physical impossibility of the upper coils passing through the lower ones.

The solution illustrates how the lower part of the slinky remains complete undisturbed until the collapsing upper part reaches it. Note, however, that we have not modeled some consequences of the collapsing coils. Upper coils impacting on lower ones will accelerate the process of collapse and advance the time of the crunch point where the slinky is completely collapsed. Thus, in reality, the collapse should happen faster than indicated in Fig. 1.
Figure 1: Height $z(\zeta, t)$ of the slinky at several times ($\tau = \sqrt{M/K}$ is the time for the signal to travel the length of the slinky). The top ($\zeta = 1$) is on the right hand side, the bottom ($\zeta = 0$) on the left. The flat segments indicate where the slinky has collapsed. The dotted line shows the solution at the initial time.
3 Conclusion

The behaviour of the falling slinky is quite surprising. Slow-motion videos confirm very convincingly that the lower portion remains completely motionless until the upper coils collide with it.

We have ignored several factors that influence the motion. An analysis that considers some of the points omitted here can be found in [1]. Clearly, there is a physical limit on the distance between successive coils and when they contact each other their dynamics changes. The physical spring can also undergo torsion, which may affect its motion. Finally, we do not discuss the motion for times greater than the ‘crunch point’ $t = \tau$ when, in theory, the slinky has reached complete collapse. In reality, it topples chaotically from this point on.

References

[3] YouTube video of falling slinky at http://www.youtube.com/user/1veritasium