The Power Tower Function

Peter Lynch
School of Mathematical Sciences, UCD, Dublin

January 2013

Abstract

We consider the function called the *power tower function*, defined by iterated exponentiation of the real variable x. It is found to converge on the interval $\exp(-e) < x < \exp(1/e)$. The function may be expressed as the inverse of the function $x = y^{1/y}$, allowing an extension of the domain to $0 < x < \exp(1/e)$. It may also be expressed in terms of the Lambert W-function, enabling an analytical continuation to the complex plane.

We study some properties of the 'power tower function', defined iteratively as the limit of the sequence of functions

$$y_1 = x \qquad y_{n+1} = x^{y_n}$$

as $n \to \infty$. It would appear at first glance that this sequence diverges for $x > 1$. In fact, the sequence converges for $x \in (e^{-e}, e^{1/e})$.

1 Definitions

We are interested in the properties of the function defined by an 'infinite tower' of exponents:

$$y(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} \tag{1}$$

Let us call this the *power tower function.* We note the convention

$$x^{x^x} \equiv x^{(x^x)} \qquad \text{and } \textit{not} \qquad x^{x^x} = (x^x)^x = x^{x^2}\,.$$

Thus, the tower is constructed *downwards*; it should really be denoted as

$$y(x) = {}^{\cdot^{\cdot^{\cdot}}} x^{x^x}$$

as each subsequent x is adjoined at the bottom of the tower.

If (1) is to have any meaning for a particular value of x, we need to show that it has a well-defined value for that x. We consider the iterative process

$$y_1 = x \qquad y_{n+1} = x^{y_n}\,. \tag{2}$$

This generates the infinite sequence of successive approximations to (1):

$$\{y_1, y_2, y_3, \ldots\} = \left\{x, x^x, x^{x^x}, \ldots\right\}$$

If the sequence converges to a value $y = y(x)$, it follows that

$$y = x^y \tag{3}$$

This gives an explicit expression for x as a function of y:

$$x = y^{1/y} \tag{4}$$

Taking the derivative of this function we get

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \left(\frac{1 - \log y}{y^2}\right) x$$

which vanishes when $\log y = 1$ or $y = e$. At this point, $x = \exp(1/e)$. Moreover, it is easily shown that $\lim_{y\to 0} x = 0$ and $\lim_{y\to\infty} x = 1$.

2 Iterative Solution

The logarithm of (3) gives us $\log y = y \log x$ or

$$y = \exp(\xi y) \tag{5}$$

where $\xi = \log x$. This is in a form suited for iterative solution: given a value of x, and therefore of ξ, we seek a value y such that the graph of $\exp(\xi y)$ intersects the diagonal line $y = y$. Starting from some value $y_{(0)}$, we compute the iterations

$$y_{(n+1)} = \exp(\xi y_{(n)}) \tag{6}$$

(see, e.g., [4], p. 315). In Fig. 1, we sketch the graph of $\exp(\xi y)$ for a selection of values of ξ. For $\xi < 0$, corresponding to $x < 1$, there is a single root of (5) (Fig. 1, top left panel). For $0 < \xi < 1/e$ (that is, for $1 < x < e^{1/e}$), there are two roots (Fig. 1, top right panel). For $\xi = 1/e$ ($x = e^{1/e}$), there is one double root (bottom left panel). Finally, for $\xi > 1/e$ ($x > e^{1/e}$), there are no roots (bottom right panel).

The iterative method converges only if the derivative of the function on the right side of (6) has modulus less than unity. Since

$$\frac{\mathrm{d}}{\mathrm{d}y} \exp(\xi y) = \xi y$$

this criterion is satisfied for $-e < \xi < 0$, and also for the smaller of the two roots when $0 < \xi < 1/e$. We therefore expect to obtain a single solution for $-e < \xi < 1/e$ or $\exp(-e) < x < \exp(1/e)$.

We plot the function (3) in Fig. 2 (left panel). It is defined for all positive y. Its derivative vanishes at $y = e$ where it takes its maximum value $\exp(1/e)$. It is monotone increasing on the interval $(0, e)$ and has an inverse function on this interval. This inverse is the power tower function (1), plotted in Fig. 2 (right panel).

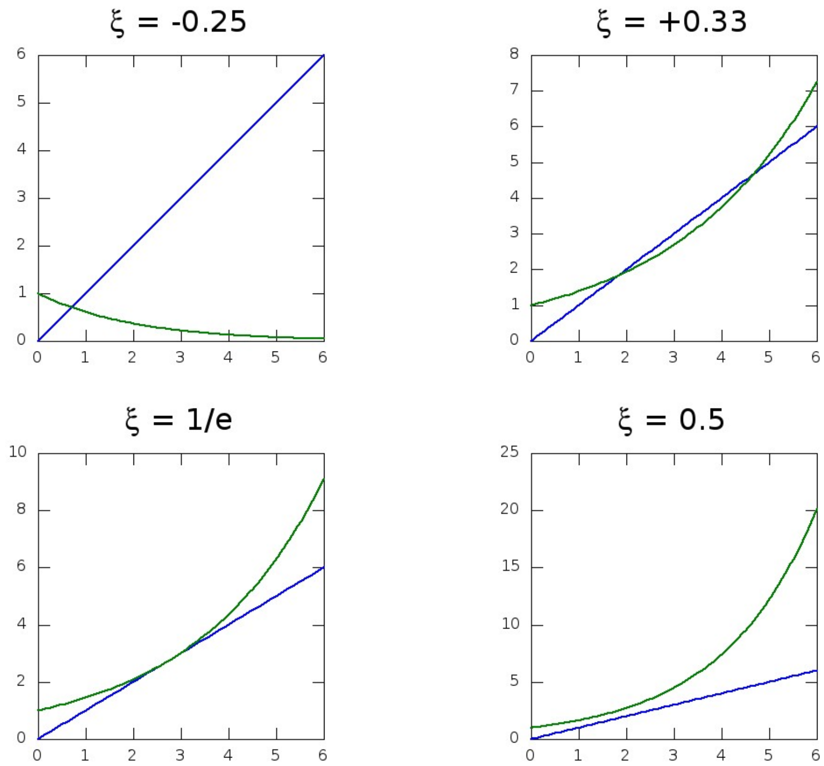

Figure 1: Graphs of $y = \exp(\xi y)$ for a range of values $\xi \in \{-0.25, +0.33, 1/e, 0.5\}$. The diagonal straight line represents $y = y$.

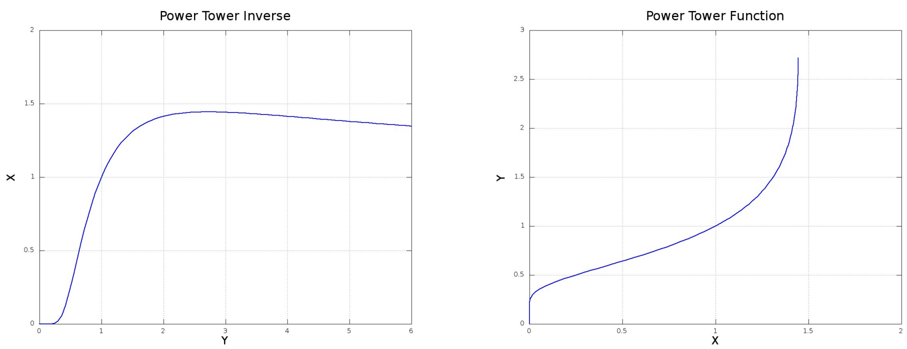

Figure 2: Left panel: $x = y^{1/y}$ for $y \in (0, 6)$. Right panel: Power tower function (1) for $x \in (0, \exp(1/e))$.

3 Johann Heinrich Lambert

Johann Heinrich Lambert (1728–1777) was a Swiss mathematician, physicist and astronomer. Lambert was born about twenty years later than Euler. In one of his papers, Euler referred to his younger compatriot as “The ingenious engineer Lambert”.

Lambert is remembered as the first person to prove the irrationality of π. Euler had earlier proved that e is irrational. Lambert conjectured that e and π were both transcendental numbers, but the proof was not found for about another century. The transcendence of e was shown in 1873 by Charles Hermite and, in 1882, Ferdinand von Lindemann published a proof that π is transcendental.

Lambert had very wide scientific interests. He introduced the hyperbolic functions into spherical geometry and proved some key results for hyperbolic triangles. He also devised several map projections that are still in use today.

Figure 3: Lambert conformal conic projection with standard parallels at 20°N and 50°N (image from Wikimedia Commons).

4 The Lambert W-function

In studying the solutions of a family of algebraic equations, Lambert introduced a power series related to a function that has proved to be of wide value and importance. The Lambert W-function is defined as the inverse of the function $z = w\exp(w)$:

$$w = W(z) \qquad \Longleftrightarrow \qquad z = w\exp(w)\,. \tag{7}$$

A plot of $w = W(z)$ is presented in Fig. 4.

We confine attention to real values of $W(z)$, which means that $z \geq -1/e$. The W-function is single-valued for $z \geq 0$ and double-valued for $-1/e < z < 0$. The constraint $W(z) > -1$ defines a single-valued function on $z \in [-1/e, +\infty)$. This is the principal branch, denoted when appropriate as $W_0(z)$. The other branch, real on $z \in [-1/e, 0)$, is denoted $W_{-1}(z)$. For further details, see [1].

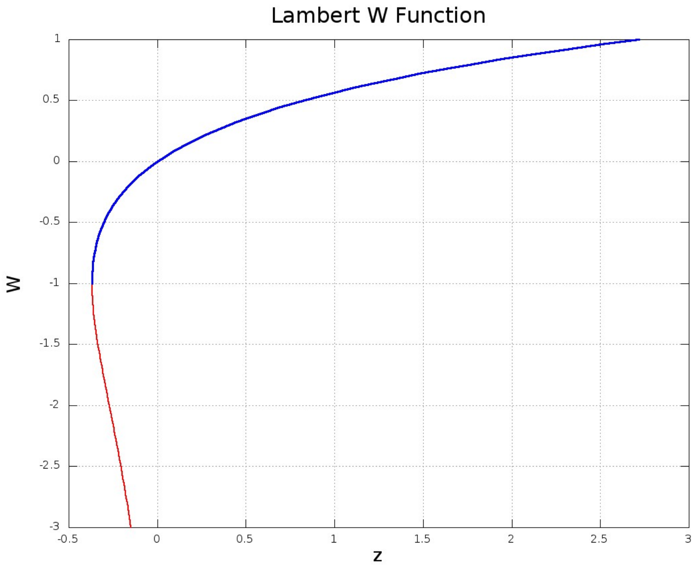

Figure 4: Lambert W-function $w = W(z)$, defined as the inverse of $z = w\exp(w)$.

Applications of the W-Function

The Lambert W-function occurs frequently in mathematics and physics. Indeed, it has been "re-discovered' several times in various contexts. In pure mathematics, the W-function is valuable in solving transcendental and differential equations, in combinatorics (as the Tree function), for delay differential equations and for iterated epxonentials (which is the context in which we have introduced it). In theoretical computer science, it is used in the analysis of algorithms. Physical applications include water waves, combustion problems, population growth, eigenstates of the hydrogen molecule and, recently, quantum gravity.

The W-function also serves as a pedagogical aid. It is a useful example in introducing implicit functions. It is also a valuable test case for numerical solution methods. In the context of complex variable theory, it is a simple example of a function with both algebraic and logarithmic singularities. Finally, it has a range of interesting asymptotic behaviours. For further references on this, see [1].

5 The Power Tower Function and W

We considered some properties of the function defined by an 'infinite tower' of exponents:

$$y(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} \,. \tag{8}$$

We called this the *Power Tower function.* We considered the iterative sequence of successive approximations to (8):

$$y_1 = x \qquad y_{n+1} = x^{y_n} \tag{9}$$

as $n \to \infty$. Through numerical experiments, we found that the sequence $\{y_n\}$ converges for $e^{-e} < x < e^{1/e}$. In fact, this result was first proved by Euler [1].

When (9) converges, we have an explicit expression for x as a function of y:

$$x = y^{1/y} \tag{10}$$

This is well defined for all positive y. Its inverse has a branch point at $(x, y) = (e^{1/e}, e)$.

Defining $\xi = \log x$, it follows that $y = \exp(\xi y)$. We can write this as

$$(-\xi y)\exp(-\xi y) = (-\xi)$$

We now define $z = -\xi$ and $w = -\xi y$ and have $z = w\exp(w)$. But, by the definition of the Lambert W-function, this means that $w = W(z)$.

Returning to variables x and y, we conclude that

$$y = \frac{W(-\log x)}{-\log x} \tag{11}$$

which is the expression for the power tower function in terms of the Lambert W-function. This enables analytical continuation of the power tower function to the complex plane.

6 Iteration for $\mathbf{x < e^{-e}}$

The iterative sequence $\{y_n(x)\}$ converges for $\exp(-e) < x < \exp(1/e)$. For $0 < x < \exp(-e)$, it does not converge, but is found to alternate between two values. denoting these by a and b, we must have

$$x^a = b \qquad \text{and} \qquad x^b = a$$

This leads to $a^a = b^b$, but we cannot conclude that $a = b$, because the function $y = x^x$ has a turning point at $x = 1/e$ (see Fig. 5, left panel).

In Fig. 5 (right panel) we show the solution of the iterative procedure (2) for $0 < x < 0.1$. When $x > \exp(-e)$, there is convergence. When $x < \exp(-e)$, the sequence alternates between two values, shown by the upper and lower curves in the figure. The central curve for $0 < x < \exp(-e)$ is the inverse of $x = y^{1/y}$. There is a pitchfork bifurcation at $x = \exp(-e)$.

The failure of the sequence $\{y_n(x)\}$ to converge for $x < \exp(-e)$ is explained in terms of the slope of the exponential function. The iterative approximation

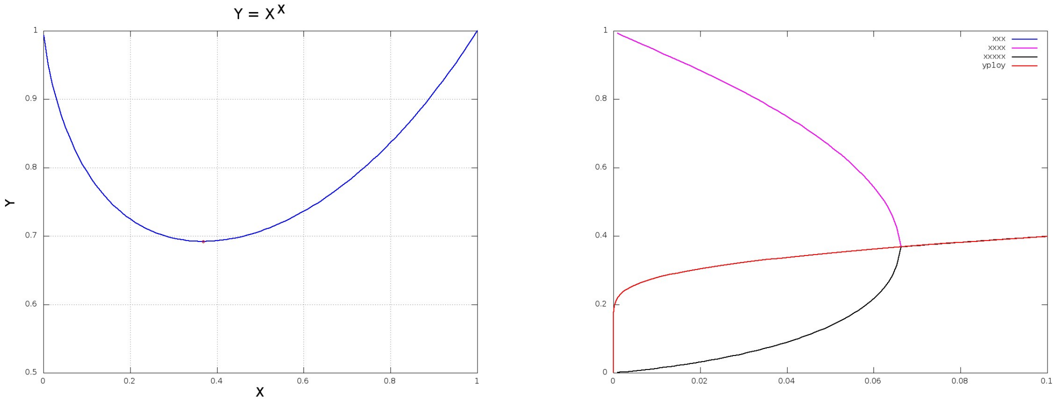

Figure 5: Left: Graph of $y = x^x$ for $0 < x < 1$. Right: Solution of the iterative procedure (2) for $0 < x < 0.1$, showing a pitchfork bifurcation at $x = \exp(-e)$.

to $y = \exp(\xi y)$ diverges if the absolute value of the gradient of the exponential exceeds 1. But the gradient is $\xi \exp(\xi y) = \xi y$ and this is -1 when $y = 1/e$ and $x = e^{-e}$. For smaller x, the process diverges.

Although the sequence $\{y_n(x)\}$ does not converge for $0 < x < \exp(-e)$, the power tower function may be defined on this interval by the process of analytic continuation. The function $x = y^{1/y}$ has a unique real value on the interval $y \in (0, e)$, so the power tower function may be defined as the inverse of this function, with domain $0 < x < \exp(1/e)$ and range $y \in (0, e)$.

7 Summary

The power tower function (1) is well defined on the domain $x \in (\exp(-e), \exp(1/e))$. It may be computed directly from (1) on this interval, or as the inverse of the function (3) for $y \in (0, e)$. The graph of the tower function is shown in Fig. 2 (right panel). It is monotone increasing with finite derivative on the interval $(0, \exp(1/e))$. The derivative is not defined at the extremities of the domain of definition.

The function $x = y^{1/y}$ tends to zero faster than any polynomial function as $y \to 0$. Correspondingly, the power tower function is 'steeper' than any inverse root of x as $x \to 0$.

The relationship (11) between the power tower function and the Lambert W-function allows us to extend the definition of the power tower function to the complex plain. The function has a logarithmic branch point at $x = 0$. The behaviour of the different branches of the W-function are described in [1].

References

[1] Corless, R. M., Gonnet, G. H., Hare, D. E. G., Jeffrey, D. J. and Knuth, D. E. (1996). On the Lambert W-function. *Adv. Comp. Math.* **5**, 329–359.

[2] NIST Digital Library of Mathematical Functions. `http://dlmf.nist.gov/`. Entry for Lambert W-function at `http://dlmf.nist.gov/4.13`

[3] Olver, Frank W. J., Daniel W. Lozier, Ronald F. Boisvert, Charles W. Clark, 2010: *NIST Handbook of Mathematical Functions*. Camb. Univ. Press. 968 pp. ISBN:9780521192255.

[4] Richards, Derek, 2002: *Advanced Mathematical Methods with Maple*. Cambridge Univ. Press, 862.